

Chapter 5 : Functions of two variables

1. Definitions.

• Activity.

- Let $f(x, y) = x + y^2 - 2$
- Find $f(1, -1) = 1 + (-1)^2 - 2 = 0$

• Definition 1.

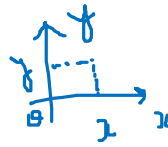
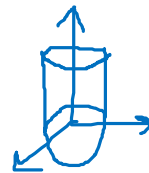
- A function of two variables has the following form:

- $f: D_f \subset \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

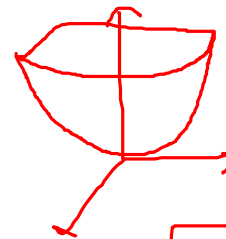
- $(x, y) \rightarrow z = f(x, y)$

- **Example.** Let $f(x, y) = \sqrt{x} + y^2 - 2$

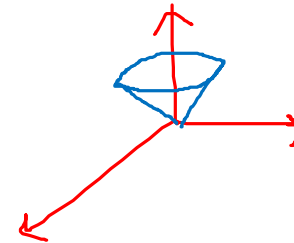
- Find $f(1, 0) = \sqrt{1} + 0 - 2 = -1$.



$$z = x^2 + y^2 = f(x, y)$$



$$z = \sqrt{x^2 + y^2}$$



Chapter 5 : Functions of two variables

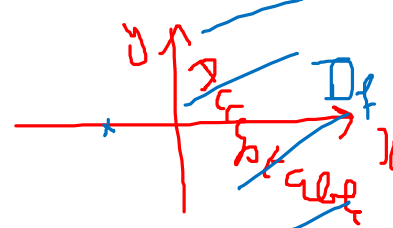
2. Domain of definition.

- **Activity.**

- Let $f(x, y) = \sqrt{x} + y^2 - 2$

- Calculate $f(-1, 1) = \nexists$ $(-1, 1) \notin D_f$

we must have, $x \geq 0$
 $D_f = \{(x, y) \in \mathbb{R}^2, x \geq 0\} = \mathbb{R}^+ \times \mathbb{R}$



- **Definition 2.**

- The domain of definition of a function f is the part of plane $\mathbb{R}^2$ such as $f(x, y)$ exists.

- **Example.** Let f be the function of two variables such as

- $f(x, y) = \frac{2x}{x-y}$; $f(5, 5) = \nexists$

- We must have

- $x - y \neq 0 \Rightarrow x \neq y$

- Thus,

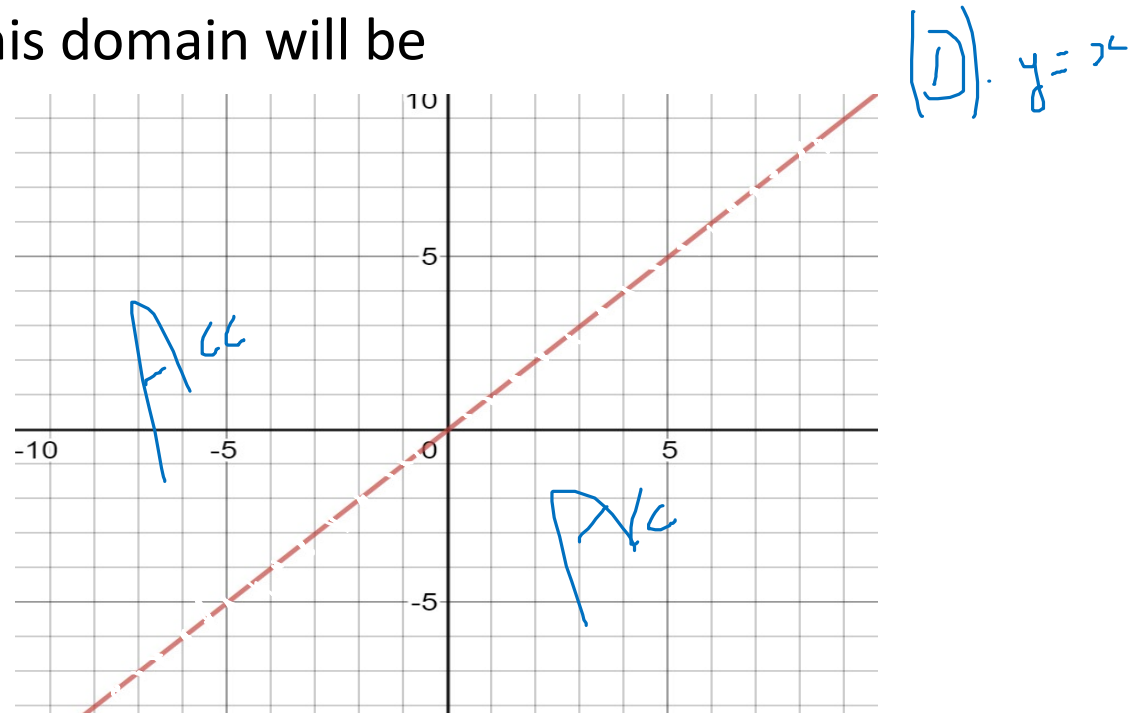
- $D_f = \{(x, y) \in \mathbb{R}^2; x \neq y\}$

Chapter 5 : Functions of two variables

2. Domaine of definition.

- We have
- $D_f = \{(x, y) \in \mathbb{R}^2; x \neq y\}$

The graphic representation of this domain will be



Chapter 5 : Functions of two variables

2. Domains of definition.

• **Example.** Let f be the function of two variables such as

• $f(x, y) = \ln(x + y)$; we must have $x + y > 0$

Then, $D_f = \{(x, y) \in \mathbb{R}^2 ; x + y > 0\}$.

• Boundaries: $x + y = 0 \Leftrightarrow y = -x$; $(D): y = -x$

• Test of Point: We choose a pt $\notin$ Boundary.
Let $A(2, 2)$ be this pt

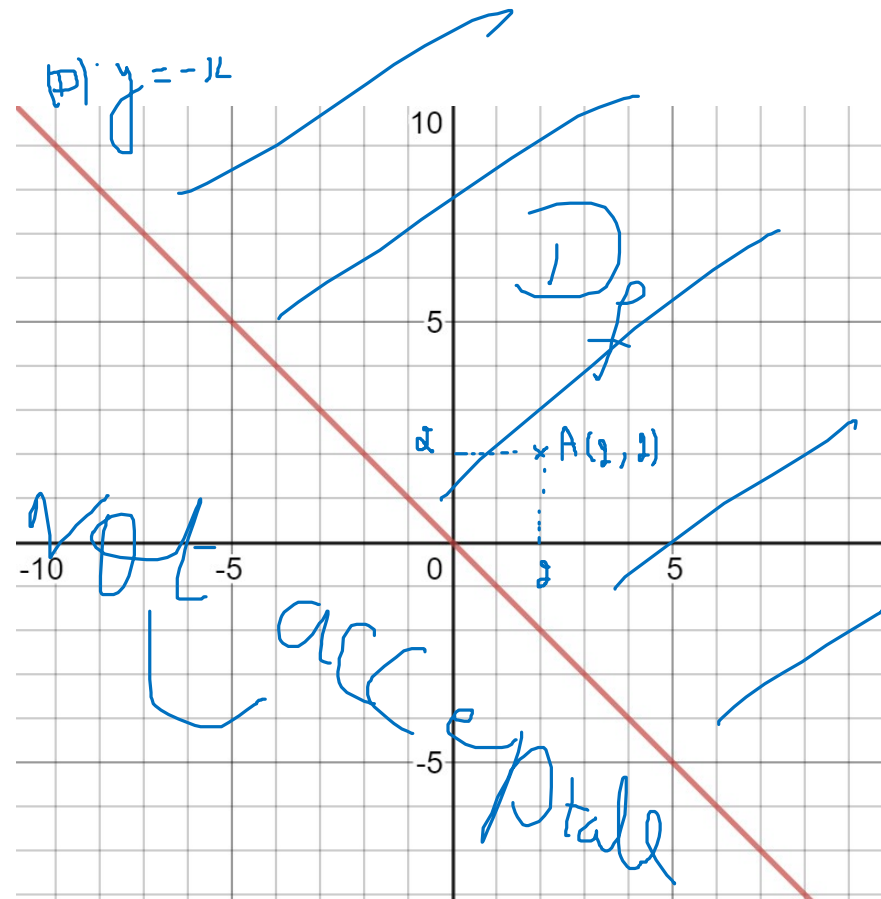
$$x + y > 0$$

$$2 + 2 > 0 ?$$

$$4 > 0 \text{ True}$$

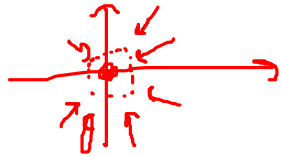
} Thus, the region of plane containing $A(2, 2)$ which is to the right of (D) , verifies $x + y > 0$ and then it is acceptable

Chapter 5 : Functions of two variables



Chapter 5 : Functions of two variables

3. Limits.

- **Introduction.** One of the big problems in the study of two-variable functions is the limit of a function f When (x, y) tends to a point (x_0, y_0) , because for it to exist it must be unique along all the possible paths that pass through the point (x_0, y_0) .
- **Example.** Study the following limit
 - $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ with $f(x, y) = \begin{cases} \frac{x^2 y}{x^3 - y^3} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$
 - 
 - One of the series passing through the point (0.0) is the set of the straight lines $y = kx$. We have:
 - $\lim_{\substack{y=kx \\ x \rightarrow 0}} f(x, kx) = \lim_{x \rightarrow 0} \frac{x^2 kx}{x^3 - (kx)^3} = \lim_{x \rightarrow 0} \frac{kx^3}{x^3(1-k^3)} = \frac{k}{1-k^3}$
 - Since the answer depends on k , then every time we change the value of k , the value of the limit changes, as a result we can conclude that the limit does not exist.

Chapter 5 : Functions of two variables

4. Continuity.

- **Definition.** A function of two variables f is said to be continuous in point (x_0, y_0) if and only if

- $f(x_0, y_0)$ exists ✓
And

- $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = f(x_0, y_0)$ ✓

- **Example.** Let f be the function defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^3 - y^3} & \text{si } (x, y) \neq (0, 0) \\ 0 & \text{si } (x, y) = (0, 0) \end{cases}$$

- Study the continuity of f in the point $(0, 0)$.

- **Solution.** We have $f(0, 0) = 0$ exists.

- We've already seen that

$$\lim_{\substack{y=kx \\ x \rightarrow 0}} f(x, kx) = \frac{k}{1-k^3} \quad \text{not unique}$$

- The limit does not exist then the fct is not continuous in $O(0, 0)$

Handwritten notes:

- $f(0, 0) = 0 \quad \exists$
- $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^3 - y^3} = \frac{k}{1-k^3}$
- $\lim \nexists$

Chapter 5 : Functions of two variables

4. Partial derivatives.

- **Activity.** Let $f(x, y) = x^2 + y^3$

- $f'_x(x, y) = 2x$

- $f'_y(x, y) = 3y^2$

$$f'_x(x, y) = 2x + 0 = 2x = \frac{\partial f}{\partial x}$$

y est

$$f'_y(x, y) = 0 + 3y^2 = \frac{\partial f}{\partial y}$$

x est

- **Introduction.** Since we study the functions of two variables, then we have to calculate the derivatives with respect to the two variables x and y and we will call them partial derivatives in relation to x and y respectively.

- By calculating partial derivatives with respect to x , we act as if the variable y is a constant, and by calculating partial derivatives with respect to y , we act as if the variable x is a constant.

-

- Notations : Let f be a function of two variables x and y .

- Partial derivative of f with respect to x is denoted $f'_x = \frac{\partial f}{\partial x}$

- Partial derivative of f with respect to y is denoted $f'_y = \frac{\partial f}{\partial y}$

Chapter 5 : Functions of two variables

4. Partial derivatives.

• **Example.** Let $f(x, y) = e^{xy}$

• $f'_x(x, y) = ye^{xy}$

• $f'_y(x, y) = xe^{xy}$

• For partial derivatives of order two, we have

• $f''_{xy} = (f'_x)'_y$

• $f''_{x^2} = (f'_x)'_x$

• $f''_{yx} = (f'_y)'_x$

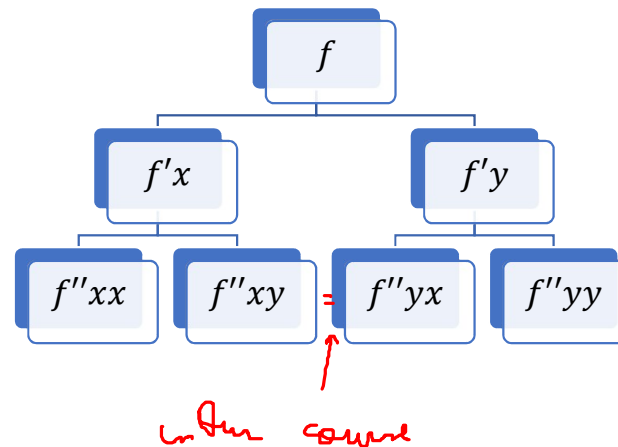
• $f''_{y^2} = (f'_y)'_y$

$$f'_{x,y}(x,y) = (xy)'_x e^{xy} = y \cdot e^{xy}$$

$y \text{ cst}$

$$f'_{y,x}(x,y) = (xy)'_y e^{xy} = x e^{xy}$$

$x \text{ cst}$



Chapter 5 : Functions of two variables

4. Partial derivatives.

- Let's take the example where $f(x, y) = x^2 + y^2$

$$x^2 + y^2$$

$$\begin{aligned}(2x)' &= 2 \\ (2y)' &= 0 + 1 = 1\end{aligned}$$

$$\bullet \quad \underbrace{f'(x, y)}_{y \text{ cst}} = 2x + 0 = 2x \quad ; \quad \underbrace{f'(x, y)}_{x \text{ cst}} = 0 + 2y = 2y$$

2nd Order:

$$f''_{xy}(x, y) = (2x)'_y = 0 \quad ; \quad f'_{xy}(x, y) = (2y)'_x = 0$$

$$f'_{yx}(x, y) = (2y)'_x = 0 \quad ; \quad f''_{yy}(x, y) = (2y)'_y = 2$$